

CENTERS OF PATH ALGEBRAS, COHN AND LEAVITT PATH ALGEBRAS

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ABSTRACT. We study the center of several types of path algebras. We start with the path algebra KE and prove that if the number of vertices is infinite then the center is zero. Otherwise, it coincides with the field K except when the graph E is a cycle in which case the center is $K[x]$, the polynomial algebra in one indeterminate. Then we compute the centers of prime Cohn and Leavitt path algebras. A lower and an upper bound for the center of a Leavitt path algebra are given by introducing the graded Baer radical for graded algebras.

1. INTRODUCTION

The work is organized as follows. We start Section 2 with some preliminaries and studying the center of the path algebra KE associated to a graph E ; then we get some implications that will be of use to determine the centers of Cohn and Leavitt path algebras. Section 3 deals with the centers of prime Cohn and Leavitt path algebras. First, we reduce the study to the finite case because we obtain that if a Cohn or Leavitt path algebra has nonzero center then the number of vertices of the underlying graph must be finite (here we are considering connected graphs). In the following step we center our attention in the 0 component of the center, which turns out to be finite dimensional, as any element it contains is proved to be symmetric. Combining this with Theorem 3 we show that if $C_K(E)$ is prime, it must be the Cohn path algebra associated to the m -petals rose graph and its center is K (Subsection 3.3). In Subsection 3.4 we study the center of prime Leavitt path algebras. Here the situation is slightly more complex and we get that the center is K when every cycle in the graph has an exit and the Laurent polynomial algebra $K[x, x^{-1}]$ in case there is a (necessarily unique) cycle without exits. In Section 4 we introduce a graded version of the Baer radical of a graded algebra; it turns out to be zero for any Leavitt path algebra. This result allows to prove that any Leavitt path algebra $L_K(E)$ is the subdirect product of a family of prime Leavitt path algebras and to conclude that the center of $L_K(E)$ is a subalgebra of a product of centers of prime Leavitt path algebras. We also find a lower bound for the center of $L_K(E)$.

2. PRELIMINARIES AND THE CENTER OF THE PATH ALGEBRA KE

We shall consider always algebras over a base field K . Let us fix some notation and terminology on graphs and algebras. A (*directed*) *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \rightarrow E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For $e \in E^1$, the vertices $s(e)$ and $r(e)$ are called the *source* and *range* of e , respectively, and e is said to be an *edge from* $s(e)$ *to* $r(e)$. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is

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called *row-finite*. A vertex which emits no edges is called a *sink*; the vertex will be called a *source* if it does not receive edges. A vertex v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set, and a *regular vertex* otherwise. Also we shall use the notation $\text{Path}(E)$ for the set of all paths of E including the vertices as trivial paths. For a path $\lambda = e_1 \cdots e_n \in \text{Path}(E)$ we will call the *length* of λ to the number n of edges appearing in λ and will denote it by $l(\lambda)$. Vertices are then paths of length 0. In this case, $s(\lambda) = s(e_1)$ and $r(\lambda) = r(e_n)$ are the *source* and *range* of λ . If μ is a path in E , and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at v* . If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists $i \in \{1, \dots, n\}$ such that $s(e) = s(e_i)$ and $e \neq e_i$.

If $E_1 = (E_1^0, E_1^1, s_1, r_1)$ and $E_2 = (E_2^0, E_2^1, s_2, r_2)$ are graphs such that $E_2^0 = E_1^0$, $E_2^1 = E_1^1$, $s_2 = r_1$ and $r_2 = s_1$, then we will say that E_2 is the *opposite* graph of E_1 . We shall use the notation $E_2 = E_1^\circ$. It is easy to see that $KE^\circ \cong (KE)^\text{op}$, the opposite algebra of KE with multiplication $x \cdot y = yx$. As a consequence there is a general principle, which we will call *duality*, stating that if P is a property that holds for the path algebra of any graph, then a “dual property” also holds in the path algebra of any graph. If E is a graph and u, v two vertices, we shall say that u and v are *connected* (denoted by $u \sim v$) if they are connected in the underlying undirected graph of E or, in other words, if there is a (finite) sequence $u = u_1, u_2, \dots, u_n = v$ such that for any i there is a path π_i such that $s(\pi_i) = u_i$, $r(\pi_i) = u_{i+1}$ or $s(\pi_i) = u_{i+1}$, $r(\pi_i) = u_i$. We shall adopt the convention that any vertex is connected to itself by a trivial path. So, one can see immediately that *connectedness* is an equivalence relation whose equivalence classes will be called *connected components* of E^0 . If we write $E^0 = \{u_i\}_i$ and consider the Peirce decomposition of the path algebra $A = KE$ given by $A = \oplus A_{uv}$, where $A_{uv} := uAv$, then $A_{uv} = 0$ when u and v are not connected. So if $E^0 = \cup_\alpha E_\alpha^0$, where the E_α^0 are the different connected components of the set of vertices, for every α we can define the ideals

$$A_\alpha = \bigoplus_{u, v \in E_\alpha^0} A_{uv}$$

and we have $A = \oplus_\alpha A_\alpha$. In this case it is trivial to check that $A_\alpha A_\beta = 0$ when $\alpha \neq \beta$ and $Z(A) = \oplus_\alpha Z(A_\alpha)$, where $Z(\cdot)$ denotes the center of the corresponding algebra. This observation means that we can restrict our attention to those algebras KE whose graph E is connected.

In the mathematical field of graph theory, the distance between two vertices in an undirected graph is the number of edges in a shortest path connecting them. This is also known as the *geodesic distance* because it is the length of the graph geodesic between those two vertices. If there is no path connecting the two vertices, that is to say, if they belong to different connected components, then conventionally the distance is defined as infinite. The vertex set (of an undirected graph) and the distance function form a metric space if and only if the graph is connected. If E is a directed graph and $u, v \in E^0$ are in the same connected component, we shall define the distance $d(u, v)$ as the geodesic distance in the underlying undirected graph associated to E . If u and v are nonconnected vertices then we shall write $d(u, v) = \infty$.

For a subset S of a vector space V we will denote by $\langle S \rangle$ the linear span of S in V .

The path algebra KE has a natural \mathbb{Z} -grading whose homogeneous components are $(KE)_n = 0$ if $n < 0$, $(KE)_0 = E^0$ and $(KE)_n$ is the linear span of the set $\{\mu \in \text{Path}(E) : l(\mu) = n\}$ for $n \geq 1$.

Let X be a set and $T: X \rightarrow X$ a map, we define $\text{Fix}(T)$ as the set $\text{Fix}(T) := \{x \in X: T(x) = x\}$.

Let A be an algebra and $x \in A$, we can define the sets $\text{ran}_A(x) = \{y \in A: xy = 0\}$ and $\text{lan}_A(x) = \{y \in A: yx = 0\}$. One easy but relevant property of path algebras is the following:

Lemma 1. *Let $A = KE$, $\mu \in \text{Path}(E)$, $u = s(\mu)$ and $v = r(\mu)$:*

- (i) $\text{ran}_A(\mu) \cap A_{vw} = 0$ for all $w \in E^0$.
- (ii) $\text{lan}_A(\mu) \cap A_{wu} = 0$ for all $w \in E^0$.

Proof. (i). Consider $x \in A_{vw}$ such that $\mu x = 0$. Write $x = \sum_i k_i \mu_i$, where $k_i \in K$ and μ_i are paths with source v and range w . We assume that the μ_i 's are all different. Then from $\mu x = 0$ we get $\sum_i k_i \mu \mu_i = 0$ and since all the paths in the set $\{\mu \mu_i\}_i$ are different we know that they are linearly independent. Therefore $k_i = 0$ for all i and so $x = 0$. The second assertion can be proved analogously. \square

Observe that for an associative algebra A with a system \mathcal{E} of orthogonal idempotents such that

$$A = \bigoplus_{u,v \in \mathcal{E}} A_{uv}, \quad A_{uv} = uAv$$

the center $Z(A)$ satisfies $Z(A) \subset \bigoplus_{u \in \mathcal{E}} A_{uu}$. Thus any central element $z \in Z(A)$ admits a decomposition $z = \sum_{u \in E^0} z_u$, where $z_u = zu \in A_{uu}$. This decomposition will be called the *Peirce decomposition* of z .

Lemma 2. *Let $z \in Z(KE) \setminus \{0\}$ and $u, v \in E^0$ such that $u \sim v$, then $zu \neq 0$ if and only if $zv \neq 0$.*

Proof. If there exists $\mu \in \text{Path}(E)$ such that $s(\mu) = u$ and $r(\mu) = v$, then note that

$$(1) \quad zu\mu = z\mu = \mu z = \mu v z.$$

If $zu = 0$ then $\mu zv = 0$, hence $zv \in \text{ran}_{KE}(\mu) \cap A_{vv} = 0$ by Lemma 1 (i). If $zv = 0$ then $zu\mu = 0$, that is, $zu \in \text{lan}_{KE}(\mu) \cap A_{uu} = 0$ by Lemma 1 (ii). Next we proof the general case: there is a finite sequence $u = u_0, u_1, \dots, u_n = v$ such that for any i there is an $f \in E^1$ such that $s(f) = u_i$ and $r(f) = u_{i+1}$ or $s(f) = u_{i+1}$ and $r(f) = u_i$. Then $zu_i \neq 0$ if and only if $zu_{i+1} \neq 0$; this proves the lemma. \square

Corollary 1. *If E is a connected graph and $Z(KE) \neq 0$, then $|E^0|$ is finite.*

Proof. Assume $z = \sum_{u \in E^0} zu \in Z(KE) \setminus \{0\}$, where only a finite number of summands is nonzero. Take a vertex u such that $zu \neq 0$. For any vertex $v \in E^0$ we have $u \sim v$ and applying Lemma 2 we get $zv \neq 0$. Since only a finite number of summands is non zero, then E^0 must be finite. \square

2.1. The center of the path algebra KE . From now on we shall assume that E has a finite number of vertices. Moreover, if E is not connected it must be a finite union of finite connected graphs E_i and KE is a direct sum of the algebras KE_i . Furthermore, $Z(KE)$ is the direct sum of the centers $Z(KE_i)$. Hence we can focus our attention on finite connected graphs.

Lemma 3. *Let $z \in Z(KE)$ with Peirce decomposition $z = \sum_{u \in E^0} z_u$. If $f \in E^1$ is such that $s(f) = u, r(f) = v$, then $z_u \in Ku$ if and only if $z_v \in Kv$.*

Proof. Assume $z_u = ku$; since $z_u f = fz_v$ we have $kf = fz_v$. If $k = 0$ then $z_u = 0$ and by Lemma 2 we get $z_v = 0$. In case $k \neq 0$ one has $\deg(kf) = 1 = \deg(fz_v)$ therefore $\deg(z_v) = 0$ and $z_v \in Kv$. Now, if $z_v = hv$ the same ideas lead one to the conclusion that $z_u \in Ku$. \square

Proposition 1. *If $u \sim v$ then $z_u \in Ku$ if and only if $z_v \in Kv$.*

Proof. We proceed by induction on the geodesic distance n between u and v . For $n = 1$ apply Lemma 3. If $n > 1$ there are vertices $u = u_1, u_2, \dots, u_n = v$ such that for any $i \in \{1, \dots, n\}$ there is an arrow from u_i to u_{i+1} or from u_{i+1} to u_i . So the induction hypothesis implies $u_{n-1} \in Ku_{n-1}$ and then by Lemma 3, $u_n = v \in Kv$. \square

As a consequence of the previous lemma for a finite connected graph E , if $z \in Z(KE)$ and its Peirce decomposition is $z = \sum_{u \in E^0} z_u$, we have the following dichotomy:

- (i) $z_u \in Ku$ for all $u \in E^0$.
- (ii) $z_u \notin Ku$ for all $u \in E^0$.

In the first case the central element is of the form $z = \sum_{u \in E^0} k_u u$, where $k_u \in K$. Next we prove that all the scalars k_u agree.

Lemma 4. *Let $z \in Z(KE) \setminus \{0\}$ with Peirce decomposition of the form $z = ku + hv + \sum_{w \neq u, v} z_w$, where $u \neq v$, $k, h \in K^\times$ and $z_w \in A_{ww}$. If u and v are connected, then $k = h$.*

Proof. Proceed by induction on the number $d = d(u, v)$. If $d = 1$ we may assume without loss in generality that there is an $f \in E^1$ such that $s(f) = u$ and $r(f) = v$. Then $zf = fz$ yields $kf = hf$, hence $k = h$. Now suppose that the property holds whenever $d < n$. Consider now two vertices u and v such that $d(u, v) = n$. There is a vertex w with $d(u, w) = 1$ and $d(w, v) = n - 1$. Then there exists $f \in E^1$ with either $s(f) = u$, $r(f) = w$ or $s(f) = w$, $r(f) = u$. In the first case, $fz = zf$ implies $fz_w = kf$ hence $f(z_w - kw) = 0$ and $z_w - kw \in \text{ran}_{KE}(f) \cap A_{ww} = 0$ by Lemma 1 (i), so $z_w = kw$. Now $z = ku + hv + kw + \sum_{u' \neq u, v, w} z_{u'}$ and applying the induction hypothesis to v and w we get $k = h$. In the second case the proof is similar by using Lemma 1 (ii). \square

Thus for a connected finite graph E , the central elements are of the form (i) $z = k \sum_{u \in E^0} u = k1$, where $k \in K$, or (ii) $z = \sum_{u \in E^0} z_u$, where $z_u \notin Ku$ for all $u \in E^0$. The elements of the form $k1$ will be called *scalars elements*. From now on, we shall investigate under which conditions the path algebra KE has nonscalar central elements.

Definition 1. Let S denote the set of all nontrivial paths of E . We define the map $F_e: S \rightarrow E^1$ given by $F_e(f_1 \dots f_n) = f_1$. We shall call this map the “*first edge*” map.

Lemma 5. *Let z be a nonscalar central element with Peirce decomposition $z = \sum_{v \in E^0} z_v$. If $0 \neq z_u = \sum_{i \in I} k_i \lambda_i$ with $k_i \in K^\times$, $\lambda_i \in \text{Path}(E)$ and $f \in E^1 \cap s^{-1}(u)$ then $f = F_e(\lambda_i)$ for each $i \in I$.*

Proof. Let $v := r(f)$ and write $z_u = \sum_{i \in I} k_i \lambda_i$ and $z_v = \sum_{j \in J} h_j \mu_j$, with $k_i, h_j \in K^\times$, $\lambda_i, \mu_j \in \text{Path}(E)$. Observe that λ_i and μ_j are nontrivial paths because z is nonscalar and by virtue of Proposition 1. Since $fz = zf$, we get $z_u f = f z_v$, that is,

$$(2) \quad \sum k_i \lambda_i f - \sum h_j f \mu_j = 0.$$

We claim that $\{\lambda_i f\} \cap \{f \mu_j\} \neq \emptyset$ because, otherwise, $\{\lambda_i f\} \cup \{f \mu_j\}$ would be linearly independent, so $k_i = 0 = h_j$ for all $i \in I, j \in J$ and therefore $z_u = 0 = z_v$, a contradiction. Next we prove that for every $i \in I$ there exists a unique $j \in J$ such that $\lambda_i f = f \mu_j$. Assume on the contrary that there exists $i_0 \in I$ such that $\lambda_{i_0} f \notin \{f \mu_j\}$. Then rewrite (2) to get $k_{i_0} \lambda_{i_0} f + \sum_{i \neq i_0} k_i \lambda_i f - \sum h_j f \mu_j = 0$. This

implies $k_{i_0} = 0$, a contradiction. Now from $\lambda_i f = f \mu_j$ we have $F_e(\lambda_i) = f$ for each $i \in I$. \square

Remark 1. Lemma 5 implies in particular that for a finite connected graph E with nonscalar center there is no bifurcations at any $u \in E^0$.

Lemma 6. *If $0 \neq Z(KE) \not\subset K \cdot 1$ then there are no sinks and no sources in E .*

Proof. By duality, it suffices to prove that there are no sinks. Assume that $v \in E^0$ is a sink. Let $z \in Z(KE) \setminus K \cdot 1$ be with Peirce decomposition $z = \sum_{u \in E^0} z_u$; then, by Proposition 1, each $z_u \notin Ku$. In particular $z_v \notin Kv$. Hence $z_v = \sum_{i \in I} k_i \mu_i \tau_i^*$ with $s(\mu_i) = s(\tau_i) = v$, being μ_i or τ_i nontrivial, which is a contradiction because $s^{-1}(v) = \emptyset$. \square

Proposition 2. *If $0 \neq Z(KE) \not\subset K \cdot 1$ then E is a cycle.*

Proof. We know that E is finite, connected, with no sinks and no sources (by Lemma 6) and without bifurcations (see Remark 1). Then it is easy to prove that E is a cycle; by Corollary 1 we may assume $|E^0| = n \in \mathbb{N} \setminus \{0\}$. The proof is clear for $n = 1$ so assume $n > 1$. Take any vertex u_1 ; since it is not a sink there is a unique edge $f_1 \in s^{-1}(u_1)$; define $u_2 = r(f_1)$. If u_1, \dots, u_{i-1} have been defined for $i < n$ let $u_i = r(f_{i-1})$, where f_{i-1} is the unique edge in $s^{-1}(u_{i-1})$. Next we prove that $u_i \notin \{u_1, \dots, u_{i-1}\}$. Suppose on the contrary $u_i = u_q$ for $1 \leq q \leq i-1$; then $u_{i+1} \in \{u_1, \dots, u_{i-1}\}$. Thus $E^0 = \{u_1, \dots, u_{i-1}\}$, which contradicts the fact that $i \leq n$. We conclude that $E^0 = \{u_1, \dots, u_n\}$ and as u_n is not a sink there is only one edge $f_n \in s^{-1}(u_n)$. Now if $r(f_n) = u_i$, with $i \geq 1$, then u_1 is a source, a contradiction. Consequently $r(f_n) = u_1$ and E is a cycle. \square

Proposition 3. *If E is a cycle then $Z(KE) \cong K[x]$, the polynomial algebra in the indeterminate x . More precisely if $E^0 = \{u_1, \dots, u_n\}$ and $E^1 = \{f_1, \dots, f_n\}$, with $s(f_i) = u_i$ for every i , $r(f_i) = u_{i+1}$ for $i = 1, \dots, n-1$, and $r(f_n) = u_1$, let*

$$\begin{aligned} c_1 &= f_1 \cdots f_n \\ c_2 &= f_2 \cdots f_n f_1 \\ &\vdots \\ c_i &= f_i \cdots f_n f_1 \cdots f_{i-1}. \end{aligned}$$

Then $Z(KE) = \{\sum_{i=1}^n p(c_i) : p(x) \in K[x]\}$ and there is an isomorphism from $Z(KE)$ to $K[x]$ such that $\sum_{i=1}^n p(c_i) \mapsto p$.

We collect the results and remarks above in the following

Theorem 1. *If E is a graph then $Z(KE)$ is the direct sum of the centers of the path algebras associated to the connected components of the graph. If E is connected and has an infinite number of vertices then $Z(KE) = 0$. If E is connected and has a finite number of vertices then $Z(KE) = K1$ except if E is a cycle; in this case $Z(KE) \cong K[x]$.*

2.2. Relationship with the center of other classes of algebras. Given a graph E we can define the *extended graph* of E as the new graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e^* : e \in E^1\}$ and the functions r' and s' are defined as $r'|_{E^1} = r$, $s'|_{E^1} = s$, $r'(e^*) = s(e)$ and $s'(e^*) = r(e)$. In this subsection we would like to establish some relationships between the center of $K\hat{E}$ and the center of certain types of algebras related to the path algebra $K\hat{E}$.

Consider the following sets of elements in $K\hat{E}$:

$$(CK1) \quad e^* e' - \delta_{e,e'} r(e) \quad \text{for all } e, e' \in E^1.$$

$$(CK2) \quad v - \sum_{\{e \in E^1 \mid s(e)=v\}} ee^* \quad \text{for every regular vertex } v \in E^0.$$

Then the *Leavitt path K -algebra* associated to the graph E , denoted $L_K(E)$, can be described as $L_K(E) = K\hat{E}/J_1$, where J_1 is the ideal generated by the elements in (CK1) and (CK2) (see, for example [3]).

In the book [3] the authors define the *Cohn path K -algebra* associated to E , denoted by $C_K(E)$, as $K\hat{E}/J_2$, where J_2 is the ideal generated by the elements in (CK1).

In both cases the described path algebras arise as quotients of the path algebra over the extended graph module an ideal, and in order to determine their centers, we may follow a similar scheme.

So, consider an ideal I of $K\hat{E}$ and the algebra $A := K\hat{E}/I$. Then there is a short exact sequence

$$(3) \quad 0 \longrightarrow I \xrightarrow{j} K\hat{E} \xrightarrow{p} A \longrightarrow 0$$

where j is the inclusion and p the canonical projection. Thus, for $I = J_1$ the algebra A is isomorphic to $L_K(E)$ and for $I = J_2$ it is isomorphic to $C_K(E)$. Observe that $J_i \cap E^1 = \emptyset$ for $i = 1, 2$.

Proposition 4. *Let E be a connected graph and I an ideal of $K\hat{E}$ such that $I \cap E^1 = \emptyset$. Define $A := K\hat{E}/I$ and consider the short exact sequence in (3). If E^0 is finite then $\langle p(E^0) \rangle \cap Z(A) = K.1$ otherwise; $\langle p(E^0) \rangle \cap Z(A) = 0$.*

Proof. Take $z \in \langle p(E^0) \rangle \cap Z(A)$. Then $z = \sum_{k \in S} l_k p(u_k)$, with $l_k \in K$, $u_k \in E^0$ and where S can be infinite but only a finite number of the l_k 's are nonzero. Let $i \neq j$; if the geodesic distance $d(u_i, u_j) = 1$ then there is an $f \in E^1$ with $s(f) = u_i$ and $r(f) = u_j$ (if necessary swap i and j). Since $p(f)z = zp(f)$ we have $p(f)z = p(f) \sum_{k \in S} l_k p(u_k) = p(\sum_{k \in S} l_k f u_k) = l_j p(f)$ and $zp(f) = \sum_{k \in S} l_k p(u_k) p(f) = l_i p(f)$. Hence $(l_i - l_j)p(f) = 0$ and $(l_i - l_j)f \in I = \text{Ker}(p)$. If $l_i \neq l_j$ then $f \in I \cap E^1 = \emptyset$, a contradiction. Thus $l_i = l_j$. Suppose now that the geodesic distance $d(u_i, u_j) = n > 1$; then there exists a vertex u_k with $d(u_i, u_k) < n$ and $d(u_j, u_k) < n$. Applying a suitable induction hypothesis we have $l_i = l_k = l_j$. As a consequence, if E^0 is finite then z is a multiple of the unit and if E^0 is infinite some scalar l_k must be zero hence all of them are zero; therefore $z = 0$. \square

Corollary 2. *If A is the Cohn path algebra or the Leavitt path algebra of a connected graph then*

$$\langle E^0 \rangle \cap Z(A) = \begin{cases} K.1 & \text{if } E^0 \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

3. CENTERS OF PRIME COHN AND LEAVITT PATH ALGEBRAS

Once we have determined the center of KE we are interested in the study of the centers of the Cohn path algebra $C_K(E)$ and of the Leavitt path algebra $L_K(E)$. A first step in the study of the center of a Leavitt path algebra was given in [4], where the authors determined the center of a simple Leavitt path algebra. The following natural step is to try to determine the center of prime Leavitt path algebras. The starting point in this section will be to prove that the existence of a nonzero center for a prime Leavitt or Cohn path algebra forces the finiteness of the number of vertices in the graph. One of the relevant tools in the theory of Cohn and of Leavitt path algebras which we shall need is the natural \mathbb{Z} -grading, for where the vertices have degree 0 and the elements of the form $\sigma\tau^*$ have degree $n - m$ for σ and τ paths of lengths n and m respectively. The degree of a homogeneous element x in a Cohn or Leavitt path algebra will be denoted by $\deg(x)$.

3.1. Reduction to the finite case. The graphs E in this subsection are not necessarily row-finite unless otherwise specified. Recall that for a graph E a subset $H \subset E^0$ is said to be *hereditary* when for any two vertices u, v such that there is a path μ with $s(\mu) = u$ and $r(\mu) = v$, if $u \in H$ then $v \in H$. A hereditary set is *saturated* if every regular vertex which feeds into H and only into H is again in H , that is, if $s^{-1}(v) \neq \emptyset$ is finite and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. We recall also that the ideal of $A = C_K(E)$ or $L_K(E)$ generated by a hereditary set H is easily seen to agree with the set of all linear combinations of elements of the form $\alpha\beta^*$, where α and β are paths such that $r(\alpha) \in H$.

Lemma 7. *If $A = C_K(E)$ or $L_K(E)$ and $\mu \in \text{Path}(E)$ with $v = r(\mu)$, then the left multiplication operator $L_\mu: A \rightarrow A$ given by $a \mapsto \mu a$ satisfies $\ker(L_\mu) \subset \text{ran}_A(v)$. Moreover, $\ker(L_\mu) \cap A_{vw} = 0$ for all $w \in E^0$.*

Proof. If $L_\mu(a) = 0$ then $\mu a = 0$; therefore $\mu^* \mu a = 0$, that is, $va = 0$ and $a \in \text{ran}_A(v)$. If $a \in A_{vw} \cap \ker(L_\mu)$ then $a \in \text{ran}_A(v)$; hence $va = a$ ($a \in A_{vw}$) or $va = 0$ ($a \in \text{ran}_A(v)$); in both cases $a = 0$. \square

Proposition 5. *Let E be any graph and $A = C_K(E)$ or $L_K(E)$. Let $z \in Z(A) \setminus \{0\}$ and consider its Peirce decomposition $z = \sum_w z_w$. Fix $u, v \in E^0$ and assume that there is some path μ with $s(\mu) = u$ and $r(\mu) = v$; then $z_u = 0$ implies $z_v = 0$.*

Proof. Since $z\mu = \mu z$ we get $z_u \mu = \mu z_v$. If $z_u = 0$ we have $0 = \mu z_v$, that is, $z_v \in \ker(L_\mu) \cap A_{vv} = 0$. \square

Corollary 3. *Under the conditions in the previous proposition the set $H := \{u \in E^0 : z_u = 0\}$ is hereditary.*

Corollary 4. *Under the conditions in the previous proposition we have $AzI(H) = 0$.*

Proof. Take $a \in A$ and $q \in I(H)$. Then $q = \sum_i l_i \alpha_i \beta_i^*$, with $l_i \in K$ and $r(\alpha_i) \in H$ for each i . Then $azq = a \sum_i l_i \alpha_i z \beta_i^* = a \sum_i l_i \alpha_i z r(\alpha_i) \beta_i^* = 0$ since $r(\alpha_i) \in H$. \square

Proposition 6. *If E is a graph and $A = C_K(E)$ or $L_K(E)$ is prime, then $Z(A) \neq 0$ implies that E^0 is finite.*

Proof. Take a nonzero central element z . Since A is prime and $Az, I(H)$ are ideals of A whose product is zero by Corollary 2, we conclude that either $Az = 0$ or $I(H) = 0$. But since $z \neq 0$ we have $Az \neq 0$. Thus $I(H) = 0$, hence $H = \emptyset$ and we conclude that $z_u \neq 0$ for any u . This forces the finiteness of E^0 since the number of nonzero components in the Peirce decomposition of z is necessarily finite. \square

Now we know that in order to have nonzero center for the prime algebras $A = C_K(E)$ or $L_K(E)$ we need finiteness of E^0 .

Remark 2. It is easy to realize that for any positive integer n there is a graph E with $|E^0| = n$ such that $L_K(E)$ is prime and with nonzero center. Indeed, consider the graph consisting of n vertices arranged in a single line. This is a simple algebra $L_K(E)$ whose center is K . However we will prove that for a prime Cohn path algebra $C_K(E)$ the number of vertices in E must be 1.

Let E be a row-finite graph and $A := C_K(E)$. Let $u \in E^0$ and suppose u is not a sink. Write $s^{-1}(u) = \{f_i : i = 1, \dots, n\}$. Then for any nontrivial path μ we have

$$(4) \quad (u - \sum_i f_i f_i^*)\mu = 0 \text{ or, equivalently, } \mu^*(u - \sum_i f_i f_i^*) = 0.$$

The element $u - \sum_i f_i f_i^*$ is an idempotent and

$$(5) \quad (u - \sum_i f_i f_i^*) A (u - \sum_i f_i f_i^*) = K(u - \sum_i f_i f_i^*).$$

This follows by (4) and taking into account $(u - \sum_i f_i f_i^*)v = \delta_{u,v}(u - \sum_i f_i f_i^*)$. Moreover, if $v \in E^0$ is a sink then

$$(6) \quad (u - \sum_i f_i f_i^*) A v = v A (u - \sum_i f_i f_i^*) = 0$$

and for any two different vertices $u, v \in E^0$ (neither of them being sinks) we have

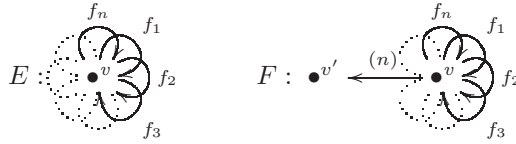
$$(7) \quad (u - \sum_i f_i f_i^*) A (v - \sum_j g_j g_j^*) = 0,$$

where $s^{-1}(u) = \{f_i\}$ and $s^{-1}(v) = \{g_j\}$.

At this point it would be convenient to recall the elementary characterization of primeness for an associative algebra: A is prime if and only if for any two elements $a, b \in A$ the fact $aAb = 0$ implies $a = 0$ or $b = 0$. At the level of Leavitt path algebras there is a purely graph-theoretic characterization of primeness. Recall that a graph E satisfies *Condition (MT3)* if for every $v, w \in E^0$ there exist $u \in E^0$ and paths $\mu, \tau \in \text{Path}(E)$ such that $s(\mu) = v$, $s(\tau) = w$ and $r(\mu) = r(\tau) = u$. In recent papers on prime Leavitt path algebras, the term “Condition (MT3)” has been replaced by the more descriptive term *downward directed*. It is proved in [7] that in the row-finite case, $L_K(E)$ is prime if and only if the graph E is downward directed.

Theorem 2. *Let E be a row-finite graph; then $C_K(E)$ is prime if and only if $|E^0| = 1$.*

Proof. Assume first that $C_K(E)$ is prime. Take $u, v \in E^0$ different. If they are not sinks then formula (7) contradicts the primeness of A . If one of them is a sink, then formula (6) contradicts also the primeness of the algebra. Finally, if u and v are sinks we have $uAv = 0$ contradicting once more the primeness of the algebra. Hence $|E^0| = 1$. Then using [3, Theorem 1.5.17] with $X = \emptyset$ the Cohn path algebra $C_K(E)$ is isomorphic to the Leavitt path algebra $L_K(F)$ which is prime because F is downward directed.



□

3.2. Some properties of the centers. In this section we shall study properties of the centers of Cohn path algebras and Leavitt path algebras that will be of interest for our purposes.

Theorem 3. *Let A be a \mathbb{Z} -graded algebra $A = \oplus A_n$ with involution $*$ such that $A_n^* = A_{-n}$ for any integer n . Let $Z := Z(A)$ be its center provided with the induced \mathbb{Z} -grading $Z = \oplus_n Z_n$, where $Z_n = Z \cap A_n$. Suppose that Z is a domain and Z_0 a field. Then each component Z_n is isomorphic (as a Z_0 -vector space) to Z_0 . Moreover either $Z = Z_0$ or there is an isomorphism $Z \cong Z_0[x, x^{-1}]$ of graded F -algebras.*

Proof. If $Z_n \neq 0$ take $0 \neq x \in Z_n$. Then $xx^* \in Z_0$. Moreover since Z is a domain $0 \neq xx^* \in Z_0$ hence x is invertible. Thus any nonzero homogeneous element is invertible. Consider now the linear map $L_x: Z_0 \rightarrow Z_n$ such that $L_x(a) = xa$. This is bijective since x is invertible and so it is an isomorphism of Z_0 vector spaces. If there is no positive integer n such that $Z_n \neq 0$ then $Z = Z_0$. Suppose on the contrary this is not true and consider the minimum positive integer n such that $Z_n \neq 0$. Then $Z = \bigoplus_{i \in \mathbb{Z}} Z_{in}$ because if there exists $k \in \mathbb{Z}$ with $in < k < (i+1)n$ and $Z_k \neq 0$ then $Z_k Z_{in}^* \subset Z_{k-in} = 0$ so $Z_k = 0$. As $Z_n = Z_0 x$, then $Z_{in} = Z_0 x^i$ and $Z = \bigoplus_{i \in \mathbb{Z}} Z_{in} = \bigoplus_{i \in \mathbb{Z}} Z_0 x^i$, so $\{x^i: i \in \mathbb{Z}\}$ is a basis of Z as a Z_0 -vector space. Therefore $Z \cong Z_0[x, x^{-1}]$. \square

Recall that an algebra A is *graded simple* if and only if $A^2 \neq 0$ and its only graded ideals are 0 and A .

Corollary 5. *Let $A = L_K(E)$ or $A = C_K(E)$ be a graded simple algebra. If $Z = Z(A) \neq 0$ then Z_0 is a field and $Z = Z_0$ or $Z = Z_0[x, x^{-1}]$ (the second possibility does not occur if A is simple).*

Proof. Any nonzero homogeneous element in the center is invertible (because the ideal generated by this element in A is graded and nonzero). In particular, Z_0 is a field. Let us prove now that Z is a domain. Take $x, y \in Z$ such that $xy = 0$ but $x, y \neq 0$. Write $x = \sum_{i \in \mathbb{Z}} x_i$, with $x_i \in Z \cap A_i$ and $y = \sum_{i \in \mathbb{Z}} y_i$, with $y_i \in Z \cap A_i$; consider $n := \max\{i: x_i \neq 0\}$ and $m := \max\{i: y_i \neq 0\}$. Since $xy = 0$ we have $x_n y_m = 0$, which is impossible because nonzero homogeneous elements are invertible. \square

Definition 2. Let A be the Cohn path algebra $C_K(E)$ or the Leavitt path algebra $L_K(E)$. Let $z \in A_0$. We say that z is *symmetric* if it is a linear combination of elements of the form $\mu\mu^*$, where $\mu \in \text{Path}(E)$.

Let $A = L_K(E)$ or $C_K(E)$ and $Z = Z(A)$. We want to show that any element in $Z_0 = Z(A)_0$ is symmetric. If $z \in Z_0$ and we write z as a linear combination of linearly independent monomials (see [3, Proposition 1.5.6 and Corollary 1.5.11]), then every nonzero monomial in z_0 is of the form $f\mu\tau^*f^*$, where $f \in E^1$ or it is a scalar multiple of a vertex. To prove this write $z = \sum_i l_i f\mu_i\tau_i^*g^* + r$, where $l_i \in K^\times$, $f \neq g$, $f, g \in E^1$, $\mu_i, \tau_i \in \text{Path}(E)$, with $\deg(\mu_i) = \deg(\tau_i)$. Here r stands for the remaining summands in z which either do not start by f or do not end by g^* (so we have $f^*rg = 0$). Then

$$0 = zf^*g = f^*zg = \sum_i l_i \mu_i \tau_i^* \text{ implying } 0 = \sum_i l_i f \mu_i \tau_i^* g^*$$

and since the monomials $\{f\mu_i\tau_i^*g^*\}$ are linearly independent we get $l_i = 0$, contradicting the fact that $l_i \in K^\times$. This proves that central elements of degree zero are linear combinations of vertices and monomials of the form $f\mu\tau^*f^*$, with $\mu, \tau \in \text{Path}(E)$.

The set $\{\sigma\mu^*: \sigma, \mu \in \text{Path}(E)\}$ is a basis for any Cohn path algebra (see [3, Proposition 1.5.6]). For each of these basic elements we define the *real degree* of $\sigma\mu^*$ (denoted by $\partial_{\mathbb{R}}(\sigma\mu^*)$) as the length of σ .

Definition 3. For any $\omega \in C_K(E)$ define $\partial_{\mathbb{R}}(\omega) = \max\{\partial_{\mathbb{R}}(\sigma_i\mu_i^*)\}$, where $\omega = \sum_{i \in I} l_i \sigma_i \mu_i^*$ is the expression of ω as a linear combination of the elements in the mentioned basis of the algebra.

We can also define the notion of real degree in the context of Leavitt path algebras by considering the basis given in [3, Corollary 1.5.11]. Since each element in this basis is of the form $\lambda\mu^*$ we can define the *real degree* of this element as the length

$l(\lambda)$. Then the *real degree* of an arbitrary element can be defined as the maximum of the real degrees of the basis elements in its expression as a linear combination of the basis.

Next we proceed to prove that any homogeneous element z of degree zero in the center is symmetric. Assume that $z = \sum_i l_i \mu f \alpha_i \beta_i^* g^* \mu^* + s + r$, where $l_i \in K$, $\mu, \alpha_i, \beta_i \in \text{Path}(E)$, $f, g \in E^1$, $f \neq g$, $\deg(\alpha_i) = \deg(\beta_i) \geq 0$, s is symmetric being all its summands of real degree $\leq n$, and r is a linear combination of walks whose real degrees are $> \deg(\mu)$ and that either do not start with μf or do not end with $g^* \mu^*$. Observe that we can assume without loss of generality that the set of walks $\{\mu f \alpha_i \beta_i^* g^* \mu^*\}$ is linearly independent.

Lemma 8. *We have $f^* \mu^* r \mu g = 0$.*

Proof. Take a summand of r , which we know it is of the form $g_1 \cdots g_q h_q^* \cdots h_1^*$ with $q > n = \deg(\mu)$. Then, if $\mu = t_1 \cdots t_n$ we have

$$f^* t_n^* \cdots t_1^* g_1 \cdots g_q h_q^* \cdots h_1^* t_1 \cdots t_n g \neq 0$$

if and only if $t_i = g_i = h_i$ for $i = 1, \dots, n$ and $g_{n+1} = f$, $h_{n+1} = g$. But this implies that the summand $g_1 \cdots g_q h_q^* \cdots h_1^*$ starts by μf and ends by $g^* \mu^*$, a contradiction. \square

Lemma 9. *We get $f^* \mu^* s \mu g = 0$.*

Proof. Consider a summand of s , say $g_1 \cdots g_q g_q^* \cdots g_1^*$ with $q \leq n$. Then, in case $n > q$ and since $\mu = t_1 \cdots t_n$, we have

$$f^* t_n^* \cdots t_1^* g_1 \cdots g_q g_q^* \cdots g_1^* t_1 \cdots t_n g = \Pi_{i=1}^q \delta_{g_i, t_i} f^* t_n^* \cdots t_{q+1}^* t_{q+1} \cdots t_n g = \Pi_{i=1}^q \delta_{g_i, t_i} f^* g = 0.$$

And if $n = q$ we get similarly $f^* t_n^* \cdots t_1^* g_1 \cdots g_q g_q^* \cdots g_1^* t_1 \cdots t_n g = 0$. \square

Theorem 4. *Every element of degree zero in the center of $C_K(E)$ or of $L_K(E)$ is symmetric.*

Proof. Take $z \in Z_0$ as before. We have $0 = f^* \mu^* \mu g z = f^* \mu^* z \mu g = \sum_i l_i \alpha_i \beta_i^*$. But the set $\{\alpha_i \beta_i^*\}$ is linearly independent because $\{\mu f \alpha_i \beta_i^* g^* \mu^*\}$ is. This implies $l_i = 0$ and therefore z is symmetric. \square

Definition 4. Let A be an algebra with involution and $a \in A$. Then we define the operator T_a as the linear map $T_a: A \rightarrow A$ given by $T_a(x) := a^* x a$.

Note that for any $a, b \in A$ we have $T_a T_b = T_{ba}$.

Lemma 10. *Let A be $L_K(E)$ or $C_K(E)$ and let $f \in E^1$ be such that $s(f) = r(f) = u$. Consider the linear map $T_f: uAu \rightarrow uAu$. Then, if ω is a fixed point of T_f of degree zero, we have $\omega = ku$ for some scalar k .*

Proof. Assume $T_f(\omega) = \omega$, then $T_f^n(\omega) = \omega$ for all n . It is easy to see that any element $h_1 \cdots h_k g_1^* \cdots g_k^*$ with $h_i, g_i \in E^1$ and some $h_i \neq f$ or some $g_j \neq f$ is in the kernel of T_f^m for suitable m . Thus ω is a linear combination of elements of the form $f^n (f^*)^m$. But since $\deg(\omega) = 0$ we have $n = m$. Now $T_f(f^n (f^*)^n) = f^{n-1} (f^*)^{n-1}$ implies that ω is a scalar multiple of u . \square

Corollary 6. *Let E be a graph. If $f \in E^1$ is such that $s(f) = r(f) = u$, take $z \in Z_0$ for $A = L_K(E)$ or $C_K(E)$. If $z = \sum z_w$ is the Peirce decomposition of z then $z_u \in Ku$.*

Proof. Since $zf = fz$ and $s(f) = r(f) = u$ we have $z_u f = f z_u$. On the other hand we have $T_f(z_u) = f^* z_u f = f^* f z_u = z_u$ and by the previous lemma we conclude $z_u \in Ku$. \square

Lemma 11. *Let A be $L_K(E)$ or $C_K(E)$ and let $c = e_1 \dots e_n \in \text{Path}(E)$ be a closed path based at u . Consider the linear map $T_c: uAu \rightarrow uAu$ given by $T_c(x) := c^*xc$. If ω is a fixed point of T_c of degree zero, then $\omega \in Ku$.*

Proof. First we show that for any element $z = h_1 \dots h_k g_k^* \dots g_1^*$ the following dichotomy holds: either there is an m such that $T_c^m(z) = 0$ or there is an m such that $T_c^m(z) = u$. To prove this, consider $z = h_1 \dots h_k g_k^* \dots g_1^*$ such that $T_c^m(z) \neq 0$ for each m . Then $0 \neq T_c(z) = e_n^* \dots e_1^* h_1 \dots h_k g_k^* \dots g_1^* e_1 \dots e_n$. If $n > k$ then $h_i = g_i = e_i$ for all $i \in \{1, \dots, k\}$ and $T_c(z) = e_n^* \dots e_{k+1}^* e_{k+1} \dots e_n = u$. If $n \leq k$, $k = qn + r$, then $T_c^{nq+1}(z) = (c^*)^{nq+1} h_1 \dots h_k g_k^* \dots g_1^* c^{nq+1} = e_n^* \dots e_{r+1}^* e_{r+1} \dots e_n = u$.

Now, consider a degree zero element ω such that $T_c(\omega) = \omega$. We can write $\omega = \sum_{i \in I} t_i \mu_i \tau_i^*$, with $t_i \in K$ and $\mu_i, \tau_i \in \text{Path}(E)$ such that $l(\mu_i) = l(\tau_i)$. Now $I = A \cup B$, where A is the set of all $i \in I$ such that $\mu_i, \tau_i^* \in \ker(T_c^m)$ for some m and $B = I \setminus A$. Since I is finite there exists an m_1 such that $T_c^{m_1}(\mu_i \tau_i^*) = 0$ for each $i \in A$. For the same reason and taking into account the proved dichotomy there is an m_2 such that $T_c^{m_2}(\mu_i \tau_i^*) = u$ for each $i \in B$. Thus, defining $m := \max(m_1, m_2)$ we have $\omega = T_c^m(\omega) = (\sum_{i \in B} t_i)u \in Ku$. \square

Corollary 7. *Let E be a graph and $A = L_K(E)$ or $C_K(E)$. Take $z \in Z_0$ with Peirce decomposition $z = \sum_{v \in E^0} z_v$. If there exists a closed path c based at a vertex u , then $z_u \in Ku$.*

Proof. Since $zc = cz$ and $s(c) = r(c) = u$ we have $z_u c = cz_u$. Moreover, $T_c(z_u) = c^* z_u c = c^* cz_u = z_u$ and by the previous lemma we conclude $z_u \in Ku$. \square

Now, suppose we are in the following situation:



where $u, v \in E^0$, c is a cycle based at v , $\mu = f_1 \dots f_k$ is a path and there is no closed path based at $s(f_i)$ for all $i \in \{1, \dots, k\}$. Take $z \in Z(A)_0$, $z = \sum_{w \in E^0} z_w$; then $\mu^* z_u \mu = \mu^* \mu z_v = z_v$ and since $z_v c = cz_v$ we have $z_v \in \text{Fix}(T_c)$, hence $z_v = lv$ for some $l \in K^\times$, but then

$$(8) \quad \mu^* z_u \mu = lv.$$

Lemma 12. *With the notation above, consider $z_u = ku + \sum_i l_i \sigma_i \sigma_i^* + \sum_j m_j \mu \gamma_j \gamma_j^* \mu^*$ with $k, l_i, m_j \in K$, $u \in E^0$, $\sigma_i, \mu, \gamma_j \in \text{Path}(E)$, $\{u\} \cup \{\sigma_i \sigma_i^*\} \cup \{\mu \gamma_j \gamma_j^* \mu^*\}$ a linear independent set and the paths σ_i not of the form $\mu \gamma_j$. Then $\sum_j m_j \mu \gamma_j \gamma_j^* \mu^* = h_0 \mu \mu^*$ for some scalar $h_0 \in K^\times$.*

Proof. From the expression for z_u we get $z_u \mu = k\mu + \sum_i l_i \sigma_i \sigma_i^* \mu + \sum_j m_j \mu \gamma_j \gamma_j^* \mu^* \mu$. It is easy to prove that if $\sigma_i \sigma_i^* \mu \neq 0$ then $\sigma_i \sigma_i^* \mu = \mu$, so $z_u \mu = k\mu + \sum_i l_i \mu + \sum_j m_j \mu \gamma_j \gamma_j^* \mu^*$ (observe that in this last sum with l_i 's we may have less summands than in the original one). Since $\mu^* z_u \mu = kv + \sum_i l_i v + \sum_j m_j \gamma_j \gamma_j^* \mu^*$ and by the formula (8) we have $\mu^* z_u \mu = kv + \sum_i l_i v + \sum_j m_j \gamma_j \gamma_j^* \mu^* = \lambda v$, which proves the lemma. \square

Proposition 7. *Let E be a finite and row-finite graph and $A = L_K(E)$ or $C_K(E)$. Then $\dim_K(Z_0)$ is finite.*

Proof. We deduce from the above lemma that if there exists a path μ such that $s(\mu) = u$ and μ connects to a cycle c then, in the expression of $z_u = ku + \sum_i l_i \sigma_i \sigma_i^* + \lambda_0 \mu \mu^*$ the cycle c does not appear. We can argue as before if we consider any other

cycle and path with source u connecting to the cycle. Since we deal with finite graphs the set of paths which may appear in the expression of z_u is finite and so $\dim_K(Z_0)$ is finite. \square

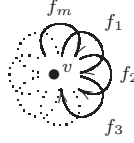
Lemma 13. *If $A = L_K(E)$ or $C_K(E)$ is prime then $Z(A)_0 = K$.*

Proof. If K is algebraically closed then $Z(A)_0$ is a finite dimensional algebra by Proposition 7 and as A is prime then $Z(A)_0$ is a domain so $Z(A)_0 = K$. If K is not algebraically closed, let Ω be the algebraic closure of K , then $Z(A_\Omega) = Z(A) \otimes \Omega$ and $Z(A_\Omega)_0 = Z(A)_0 \otimes \Omega$. Besides, if $A = L_K(E)$, then $A_\Omega = A \otimes_K \Omega = L_K(E) \otimes \Omega \cong L_\Omega(E)$ hence A_Ω is prime (because the primeness condition is given by a property of the graph). If $A = C_K(E)$ then $A_\Omega = A \otimes_K \Omega = C_K(E) \otimes \Omega \cong C_\Omega(E)$ hence A_Ω is also prime as before. In any case $1 = \dim_\Omega(Z(A_\Omega))_0 = \dim_K(Z(A)_0)$ so that $Z(A)_0 = K$. \square

3.3. The center of a prime Cohn path algebra. Next we will study the center of a prime Cohn path algebra $C_K(E)$.

Proposition 8. *If $C_K(E)$ is prime then E is the m -petals rose graph and $Z(C_K(E)) = K$.*

Proof. The primeness of $C_K(E)$ and the finiteness of E imply that $Z(C_K(E))$ is a domain. By Lemma 13 we know that $Z(C_K(E))_0 = K$ is a field. So we can apply Theorem 3 to $C_K(E)$ with its standard \mathbb{Z} -grading and involution. We conclude that $Z \cong K$ or $Z \cong K[x, x^{-1}]$. Now we must discard the second possibility. By Theorem 2 the graph E must be the m -petals rose R_m .



If $m = 0$ then $C_K(R_0) = Kv$ and there is nothing to prove. So we assume $m \geq 1$. Let $z \in Z_n$ with $n > 0$, then $z = l_0\mu_0 + l_1\mu_1(\gamma_1)^* + l_2\mu_2(\gamma_2)^* + \cdots + l_r\mu_r(\gamma_r)^*$, with $l_i \in K$, $l_r \in K^\times$, $\mu_i, \gamma_i \in \text{Path}(E)$, $\partial_{\mathbb{R}}(\mu_0) = n$, $\partial_{\mathbb{R}}(\mu_1) = n+1, \dots, \partial_{\mathbb{R}}(\mu_j) = n+j$, and $\partial_{\mathbb{R}}(\gamma_1) = 1, \dots, \partial_{\mathbb{R}}(\gamma_j) = j$. We have

$$\mu_0^* z = l_0 + l_1\mu_0^*\mu_1(\gamma_1)^* + l_2\mu_0^*\mu_2(\gamma_2)^* + \cdots + l_r\mu_0^*\mu_r(\gamma_r)^*,$$

where $\partial_{\mathbb{R}}(\mu_0^* z) \leq r$. On the other hand

$$z\mu_0^* = l_0\mu_0\mu_0^* + l_1\mu_1(\gamma_1)^*\mu_0^* + l_2\mu_2(\gamma_2)^*\mu_0^* + \cdots + l_r\mu_r(\gamma_r)^*\mu_0^*$$

and now $\max(\partial_{\mathbb{R}}(z\mu_0^*)) = n+r$, whence $n+r \leq r$, which implies $n \leq 0$, a contradiction, therefore $z = 0$. For $z \in Z_{-n}$, with $n > 0$, we may apply the involution to get again $z = 0$. Thus we conclude $Z = Z_0 = K$. \square

3.4. The center of a Prime Leavitt path algebra. We begin this subsection by introducing some definitions. For a path $\mu = e_1 \dots e_n$ we denote by μ^0 the set of vertices given by $\mu^0 := \{s(e_i) : i = 1, \dots, n\} \cup \{r(e_n)\}$.

For X a nonempty subset of an algebra A , we denote by $I(X)$ the ideal generated in A by X .

For a graph E we will denote by $P_c(E)$ the set of vertices given by $P_c(E) := \cup \mu^0$, where μ ranges in the set of all cycles without exits of the graph. Recall also the so called Condition (L): we say that a graph E satisfies *Condition (L)* if each cycle in E has an exit.

The proof of the following result is contained in [2, Proposition 3.5].

Proposition 9. *Let c and d be cycles without exits in a graph E . Then:*

- (i) $I(c^0) \cong \oplus_i M_{n_i}(K[x, x^{-1}])$.
- (ii) $I(c^0)I(d^0) = 0$.

Theorem 5 (Reduction Theorem). ([5, Proposition 3.1]). *Let E be an arbitrary graph. Then for every nonzero element $z \in L_K(E)$ there exist $\mu, \nu \in \text{Path}(E)$ such that:*

- (i) $\mu^* z \nu = kv$ for some $k \in K \setminus \{0\}$ and $v \in E^0$, or
- (ii) *there exists a vertex $w \in P_c(E)$ such that $\mu^* z \nu$ is a nonzero polynomial $p(c, c^*)$, where $p(x, x^{-1}) \in K[x, x^{-1}]$.*

Both cases are not mutually exclusive.

Theorem 6. *Let E be a graph such that $L_K(E)$ is a prime Leavitt path algebra, then $Z(L_K(E)) \neq 0$ if and only if $|E^0| < \infty$. In this case:*

- (i) $Z(L_K(E)) \cong K$ if and only if E satisfies Condition (L).
- (ii) $Z(L_K(E)) \cong K[x, x^{-1}]$ if and only if E contains a unique cycle without exits.

Proof. By Proposition 6, $Z(L_K(E)) \neq 0$ if and only if $|E^0| < \infty$. Now suppose $|E^0| < \infty$. Applying Theorem 3 and Lemma 13, we know that the center of $L_K(E)$ is isomorphic to K or to $K[x, x^{-1}]$. Suppose first that $Z(L_K(E)) \cong K$. We will see that E satisfies Condition (L). Suppose otherwise that there exists a cycle c in E without exits. Let I be the (graded) ideal generated by c^0 . By condition (i) in Proposition 9 the ideal I is isomorphic to $\oplus_i M_{n_i}(K[x, x^{-1}])$. Since $L_K(E)$ is a prime algebra and the ideals of the ideals of $L_K(E)$ are ideals of $L_K(E)$ (see, for example [10, Lemma 3.21]), the direct sum has only one term. Now, use that $L_K(E)$ is, in particular, semiprime, and [8, Remark 3.4] to get $Z(I) = Z(L_K(E)) \cap I$. Observe that $I \cong M_n(K[x, x^{-1}])$ implies $Z(I)$ isomorphic to $K[x, x^{-1}]$, which is infinite dimensional as a K -vector space. On the other hand, being $Z(L_K(E)) \cong K$ implies $Z(L_K(E)) \cap I$ is 1-dimensional over K , a contradiction. Now, suppose $Z(L_K(E)) \cong K[x, x^{-1}]$, and let φ be an isomorphism from $Z(L_K(E))$ into $K[x, x^{-1}]$. Take $a \in Z(L_K(E))$ such that $\varphi(a) = x$; by the proof of Theorem 3, we may suppose that a is a homogeneous element of degree $d > 0$. Then:

$$(\dagger) \quad aa^* = a^*a = 1.$$

We claim that there exists a vertex v in E^0 and $k \in K^\times$ such that kva or kva^* is a power of a cycle without exits. Indeed, by the Reduction Theorem there exist paths μ, ν such that $\mu^* a \nu = kv$ for some $k \in K \setminus \{0\}$ and $v \in E^0$, or $\mu^* a \nu$ is a nonzero polynomial $p(c, c^*)$, for $p(x, x^{-1}) \in K[x, x^{-1}]$. In the first case, $\mu^* a \nu = kv$; apply that a is in the center of $L_K(E)$ to get: $a\mu^* \nu = kv$. Multiply by a^* on the left hand side of each term of this identity and apply (\dagger) to get: $\mu^* \nu = kva^*$ (note that $a^* \in Z(L_K(E))$). Note that since the degree of a is $d > 0$, $\mu^* \nu$ or $(\mu^* \nu)^*$ is a path. In the second case, we apply the involution to the identity $\mu^* \nu = kva^*$ and work with $\nu^* \nu = kva$, which is a path. Hence, suppose that $(\mu^* \nu)^*$ is a path. Since $a^* \in Z(L_K(E))$, it is a closed path starting and ending by v , say $(\mu^* \nu)^* = e_1 \dots e_n$. We claim this closed path has no exits. Suppose on the contrary that there exists $i \in \{1, \dots, n\}$ and $f \in E^1$ such that $s(f) = s(e_i)$ and $f \neq e_i$. Denote by w the range of e_i . Then:

$$\begin{aligned} e_i \dots e_n e_1 \dots e_{i-1} &= (e_{i-1}^* \dots e_1^*) e_1 \dots e_{i-1} e_i \dots e_n (e_1 \dots e_{i-1}) = \\ &= (e_{i-1}^* \dots e_1^*) kva^* (e_1 \dots e_{i-1}) = kva^*. \end{aligned}$$

This implies $0 = f^* e_i \dots e_n e_1 \dots e_{i-1} = f^* (kva^*) = kf^* a^*$. Multiply by a and use again (\dagger) . We get $0 = kf^*$, a contradiction, and we have proved that kva^* is a cycle without exits. Now, suppose we are in the second case, that is, $\mu^* a \nu$ is a nonzero

polynomial $p(c, c^*)$. Since a is homogeneous, $p(c, c^*) = k'c^m$ for a nonzero $k' \in K$ and m a nonzero integer. We may suppose $m > 0$ because otherwise we would work with the identity $(\mu^*av)^* = k'c^{-m}$. Denote by $u = s(c)$. Then $\mu^*avc^{-n} = k'v$, that is, $a\mu^*vc^{-n} = k'v$. Multiply by a^* and use (\dagger) to get $\mu^*vc^{-n} = k'va^*$. Note that since $k'va^*$ is nonzero, μ^*vc^{-n} is nonzero. On the other hand, μ^*vc^{-n} is a closed path starting and ending by v . Since c has no exits, $\mu^*vc^{-n} = c^r$ for some integer r , and we have proved that $k'va^*$ is a power of a cycle without exits. To conclude the proof we show that there exists a unique cycle without exits. If c and d were two cycles without exits, then $I(c^0)I(d^0) = 0$ by condition (ii) in Proposition 9, a contradiction since $L_K(E)$ is a prime algebra. \square

4. CONSEQUENCES FOR THE CENTER OF A GENERAL LEAVITT PATH ALGEBRA

Once we know how to compute the center of a prime Leavitt path algebra, we would like to get an idea on the structure of the center of a general Leavitt path algebra associated to a row-finite graph E . In order to do this, we will give a structure theorem for Leavitt path algebras in terms of subdirect products of prime Leavitt path algebras. The key notion will be that of graded Baer radical whose behaviour is quite similar to that of the Baer radical (in a nongraded sense).

In this section we shall work with \mathbb{Z} -graded algebras and since no other grading group will be considered, the term “graded algebra” will mean \mathbb{Z} -graded algebra. Let A be a graded algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$. Recall that A is said to be *graded semiprime* when for any graded ideal I of A we have

$$I^2 = 0 \text{ implies } I = 0.$$

Also A is said to be *graded prime* when for any two graded ideals I and J of A we have

$$IJ = 0 \text{ implies } I = 0 \text{ or } J = 0.$$

It is easy to prove that A is graded semiprime if and only if for any homogeneous element $x \in A$ we have that $xAx = 0$ implies $x = 0$. It is also straightforward to see that A is graded prime if and only if for any two homogeneous elements $x, y \in A$ we have that $xAy = 0$ implies $x = 0$ or $y = 0$. As a corollary we have:

A is prime if and only if A is graded prime,

A is semiprime if and only if A is graded semiprime.

The proofs of these facts can be seen in [9, Proposition II.1.4 (1)].

Recall that a *graded prime ideal* I of A is graded ideal such that A/I is a graded prime algebra. In the same way we can define the notion of graded semiprime ideal.

Definition 5. Let A be a graded algebra; we define the *graded Baer radical* of A as the intersection of all graded prime ideals of A . We shall denote it by $\mathfrak{B}_{\text{gr}}(A)$.

The following theorem is the graded version of a well-known result on the classical Baer radical. Its proof runs parallel to that of the classical theorem.

Theorem 7. *The graded Baer radical of a graded algebra A is a graded semiprime ideal. In fact, it is the least graded ideal which is semiprime.*

Proof. If $x \in A$ is such that $xAx \subset \mathfrak{B}_{\text{gr}}(A)$ then $xAx \subset P$ for any graded prime ideal P . Since P is semiprime we have $x \in P$ hence $x \in \mathfrak{B}_{\text{gr}}(A)$. To prove that $\mathfrak{B}_{\text{gr}}(A)$ is contained in any graded semiprime ideal I we consider the class $\{P_\alpha\}_\alpha$ of all graded prime ideals of A which contain I . Next we prove that $I = \bigcap_\alpha P_\alpha$. If there is some $x \in \bigcap_\alpha P_\alpha$ such that $x \notin I$ we define $x_0 := x$. Then $x_0Ax_0 \not\subset I$ (because I is semiprime) and $x_0Ax_0 \subset \bigcap_\alpha P_\alpha$. Thus we can take some $x_1 \in x_0Ax_0$ such that $x_1 \notin I$, $x_1 \in P_\alpha$ (for any α). We repeat this argument to obtain a sequence

$X := \{x_0, x_1, \dots, x_n, x_{n+1}, \dots\}$ such that $x_n \notin I$, $x_n \in \cap_{\alpha} P_{\alpha}$ and $x_{n+1} \in x_n A x_n$. Define the family \mathfrak{F} of all graded ideals J of A such that $I \subset J$ and $J \cap X = \emptyset$. Since $I \in \mathfrak{F}$ we have $\mathfrak{F} \neq \emptyset$. Thus by Zorn's Lemma we have a maximal element $P \in \mathfrak{F}$. Let us prove that P is a prime ideal. In order to do that take two graded ideals J_i of A such that $P \subset J_i$ for $i = 1, 2$ and $J_1 J_2 \subset P$. We have to prove that $P = J_i$ for some $i = 1, 2$. Suppose on the contrary $P \subsetneq J_i$; this implies $J_i \cap X \neq \emptyset$ and so some $x_n \in J_1$ and some $x_m \in J_2$. Thus, if $l \geq \max(m, n)$ we have $x_l \in J_1 \cap J_2$ and $x_{j+1} \in J_1 J_2 \subset P$. But $P \in \mathfrak{F}$, which is a contradiction. So we have proved that P is a prime ideal, hence it is a graded prime ideal containing I . Since each $x_n \in \cap_{\alpha} P_{\alpha} \subset P$ we have $x_n \in P$, which contradicts the fact that $P \cap X = \emptyset$. This proves the theorem. \square

Definition 6. Let A be a graded algebra then we denote by \mathfrak{S} the family of all graded prime ideals of A .

Corollary 8. Let $L_K(E)$ be a Leavitt path algebra. Then

- (i) $\mathfrak{B}_{gr}(L_K(E)) = 0$.
- (ii) $L_K(E)$ is a subdirect product of prime Leavitt path algebras.

Proof. Given that $A := L_K(E)$ is graded semiprime, 0 is a graded semiprime ideal and of course it is the least graded ideal which is semiprime, so $\mathfrak{B}_{gr}(A) = 0$. To prove the second assertion, take into account that $\cap_{P \in \mathfrak{S}} P = \mathfrak{B}_{gr}(A) = 0$. Then the map

$$\begin{aligned} j: A &\rightarrow \prod_{P \in \mathfrak{S}} A/P \\ a &\mapsto (a + P)_P \end{aligned}$$

is a monomorphism. For any $Q \in \mathfrak{S}$ let $\pi_Q: \prod_P A/P \rightarrow A/Q$ be the canonical projection. Since the composition $\pi_Q j$ is an epimorphism then A is the subdirect product of the A/P , which are graded prime algebras. Moreover, since each P is a graded ideal then it is the ideal generated by some hereditary and saturated subset H_P in E^0 (see [6, Lemma 2.1 and Remark 2.2]). Therefore $A/P \cong L_K(E/H_P)$, where E/H_P denotes the quotient graph (see [6, Lemma 2.3 (1)]). So A is the subdirect product of the Leavitt path algebras $L_K(E/H_P)$. \square

Take now A to be the Leavitt path algebra $L_K(E)$. As it is well known, the graded prime ideals $P \in \mathfrak{S}$ are of the form $P = I(H)$ for a unique $H \in \mathcal{H}_E$. The quotient algebra $A/P \cong L_K(E/H)$ is prime, hence the quotient graph E/H is downward directed. Therefore there are two excluding possibilities for the graph E/H : either it satisfies condition (L) or it has a unique cycle without exits. Thus we can classify the prime ideals $P \in \mathfrak{S}$ into two flavours:

$$\mathcal{I} = \{I(H): H \in \mathcal{H}_E, E/H \text{ is downward directed and satisfies Condition (L)}\},$$

$$\mathcal{J} = \{I(H): H \in \mathcal{H}_E, E/H \text{ is downward directed and has a unique cycle without exits}\}.$$

Theorem 8. For a row-finite graph E , the center of $L_K(E)$ is a subalgebra of $\prod_{P \in \mathcal{I}} K_P \times \prod_{Q \in \mathcal{J}} K_Q[x, x^{-1}]$ containing the ideal $\bigoplus_{P \in \mathfrak{S}} Z(W_P)$, i.e.:

$$\bigoplus_{P \in \mathfrak{S}} Z(W_P) \triangleleft Z(L_K(E)) \subset \prod_{P \in \mathcal{I}} K_P \times \prod_{Q \in \mathcal{J}} K_Q[x, x^{-1}],$$

where:

- (i) $K_P = K_Q = K$ for any $P \in \mathcal{I}$ and $Q \in \mathcal{J}$.
- (ii) For any $P \in \mathfrak{S}$ the ideal W_P is defined as the intersection of all the graded prime ideals others than P .

Proof. Let A be as before the Leavitt path algebra $L_K(E)$. To prove that $Z(A)$ is a subalgebra of $\prod_{P \in \mathcal{I}} K_P \times \prod_{Q \in \mathcal{J}} K_Q[x, x^{-1}]$, consider $z \in Z(A)$ and let j and π_Q be as in the proof of Corollary 4. We want to show that $j(a)$ is in the center of $\prod_P A/P$. Since $\pi_Q j$ is an epimorphism for any Q , we have $\pi_Q j(a) \in Z(A/Q)$, so $a + Q \in Z(A/Q)$. Consequently, $j(a) \in Z(\prod_P A/P)$, which, up to isomorphism, is of the form $\prod_{P \in \mathcal{I}} K_P \times \prod_{Q \in \mathcal{J}} K_Q[x, x^{-1}]$ by Theorem 6. In order to prove the second assertion, we prove that the sum of the ideals W_P is direct. Since $W_P = \cap_{Q \in \mathfrak{S} \setminus \{P\}} Q$ we have $W_P \subset Q$ for any $Q \in \mathfrak{S}$, $Q \neq P$. So, for any $P \in \mathfrak{S}$ we have $W_P \cap (\sum_{Q \neq P} W_Q) \subset (\cap_{R \in \mathfrak{S} \setminus \{P\}} R) \cap P = \mathfrak{B}_{\text{gr}}(L_K(E)) = 0$. To finish the proof take into account that $Z(\oplus_P W_P) = \oplus_P Z(W_P)$ and that the center of an ideal of a semiprime algebra is contained in the center of the algebra. \square

The upper bound for $Z(L_K(E))$ given in Theorem 8 and Theorem 6 allows to say that the building blocks for this upper bound are K and $K[x, x^{-1}]$. The number of K 's and $K[x, x^{-1}]$'s appearing is completely determined by the cardinal of the sets \mathcal{I} and \mathcal{J} . Thus it is easily computable directly from the graph E . On the other hand the lower bound is also algorithmically computable for a given finite graph since each W_P is an intersection of ideals generated by hereditary and saturated subsets of the graph (hence each W_P is also the ideal generated by some hereditary and saturated set which can be determined from the graph). So again the ideals W_P are Leavitt path algebras. We have checked with several examples of concrete graphs that the center of a Leavitt path algebra may agree with the lower or with the upper bound described in Theorem 8. However our approach, the precise structure of the center remains an open question.

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